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Towards a General Definition for Spectroscopic Resolution

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Abstract. Judged by their instrumental profiles, spectrometers fall into two basic classes—Lorentzian and Gaussian—with many other line profile functions (sinc^m functions, Voigt functions, Airy functions, etc.) falling into one of these two categories in some limit. We demonstrate that the Rayleigh, Sparrow, and Houston resolution criteria are of limited use compared to the "equivalent width" of the line profile.

1. Introduction

Modern day spectrographs ultimately rely on the interference of a finite number of beams that traverse different optical paths to form a signal (Bell 1972). The spectrometer disperses the incoming light into a finite number of wavelength (energy) intervals, where the size of the resolution element ($\delta\lambda$) is set by the bandwidth limit imposed by the dispersing element. Different dispersive techniques produce a variety of instrumental profiles. A long-slit spectrometer in the diffraction limit produces a sinc^2 wavelength response, a property shared with acousto-optic filters. In practice, optical and mechanical defects within either device tend to make the instrumental response more Gaussian in form. The response of the Fourier Transform Spectrometer is fundamentally the sinc function, although this response is commonly apodized to produce a profile with better side-lobe behavior. An internally reflecting cavity (e.g., Fabry-Perot filter) generates an instrumental response given by the periodic Airy function. In the limit of high finesse (the periodic interval $\Delta\lambda$ divided by the line FWHM $\delta\lambda$), the Airy function reduces to the Lorentzian function.

2. Resolution Criteria

The resolution element $\delta\lambda$, or more formally the *spectral purity*, is the smallest measurable wavelength difference at a given wavelength λ. In the case of rectangular and triangular functions, the (average) instrumental width is unambiguous; for more complex functions, a characteristic width can be more difficult to define.

	$f(x)$	Area	Sparrow
G	$\exp(-\ln 16x^2/\delta x^2)$	$\left(\frac{\delta x}{2}\right)\sqrt{\frac{\pi}{\ln 2}}$	$\frac{\delta x}{\sqrt{2\ln 2}}$
L	$\frac{1}{1+(2x/\delta x)^2}$	$\left(\frac{\delta x}{2}\right)\pi$	$\frac{\delta x}{\sqrt{3}}$
A	$\frac{1}{1+\alpha\sin^2(\pi x/\Delta x)}$	$\frac{\Delta x}{\sqrt{1+\alpha}}$	$\lim\limits_{\alpha\to\infty}\frac{\delta x}{\sqrt{3}}$
S	$\mathrm{sinc}^2\left(\frac{2.7831x}{\delta x}\right)$	$\left(\frac{\pi}{2.7831}\right)\delta x$	$0.9364\,\delta x$
V_1	$\frac{a}{\pi}\int\limits_{-\infty}^{+\infty}\frac{\exp(-y^2)}{\left(x\sqrt{\ln 16}/\delta x - y\right)^2+a^2}dy$	$\lim\limits_{a\to 0}\left(\frac{\delta x}{2}\right)\sqrt{\frac{\pi}{\ln 2}}$	$\lim\limits_{a\to 0}\frac{\delta x}{\sqrt{2\ln 2}}$
V_2	$\frac{a^2}{\sqrt{\pi}}\int\limits_{-\infty}^{+\infty}\frac{\exp(-y^2)}{\left(2x/\delta x - y\right)^2+a^2}dy$	$\lim\limits_{a\to\infty}\left(\frac{\delta x}{2}\right)\frac{\sqrt{\pi}}{a}$	$\lim\limits_{a\to\infty}\frac{\delta x}{\sqrt{3}}$

Table 1. Resolution criteria for common spectral line functions in terms of the FWHM δx (see Figure 1). The Airy function is periodic over the interval Δx.

Rayleigh criterion. Lord Rayleigh (1879) first derived the resolved distance of two identical, diffraction-limited point sources with the aid of Bessel functions. This separation arises when the peak of one Bessel function falls on the first zero point of the other function. The often quoted resolution criterion, $1.220 f\lambda/L$, where f is the focal ratio of the imaging system, was only intended for use in this context. Thus, we do not investigate this criterion further.

Houston criterion. The usual metric in astronomy is to adopt the "full width at half maximum" (FWHM) as a suitable definition of spectral purity $\delta\lambda$. Houston (1926) ventured that this property can be used to define the natural separation of two identical lines which are resolved from each other.

Sparrow criterion. Sparrow (1916) suggested a clever alternative, which depends on the property of the summed line profiles. As we move the lines closer together from far apart, a minimum develops. Sparrow suggested that a natural definition for resolution results at the line separation where the saddle point first develops (i.e., the gradient at the peak of the summed profile is zero). More formally, for an instrumental response given by $f(x)$, then two sources are resolved at a separation of σ_L (Sparrow limit) when both of the following conditions are satisfied:

$$\frac{d}{dx}[f(x)+f(x+\sigma_L)] = 0,$$

$$\frac{d^2}{dx^2}[f(x)+f(x+\sigma_L)] = 0.$$

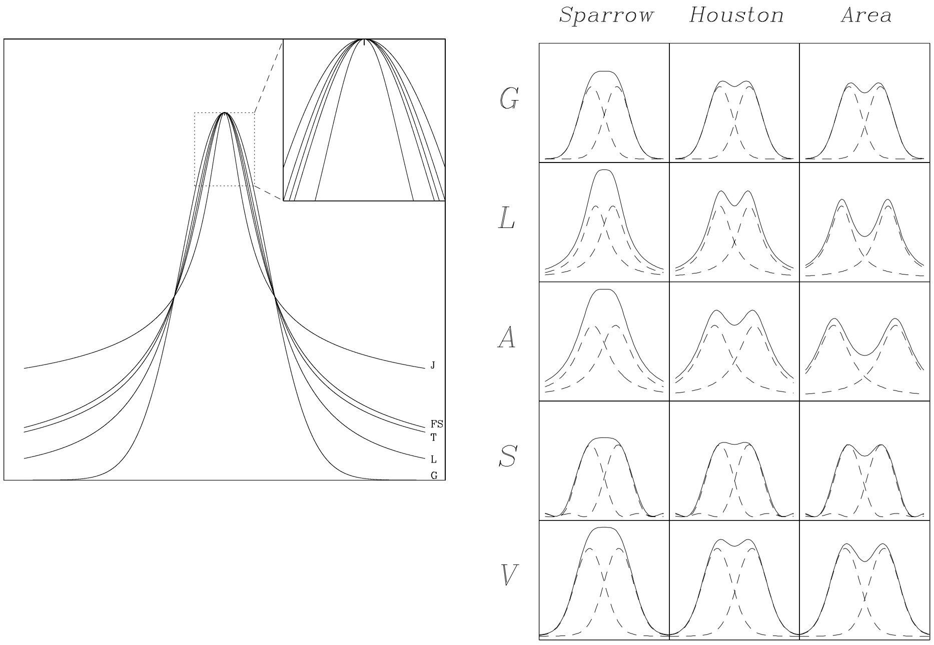

Figure 1. (a) Five functions with the same FWHM that demonstrate a gradual trend in the ratio of core power to wing power: (G) Gaussian, (L) Lorentzian, (T) $x^{-1} \tanh x$, (FS) Fraser-Suzuki function, (J) the most extreme case to date discovered by AWJ. All of these functions have continuous higher derivatives. In order for the function to remain everywhere continuous, the neighborhood of the peak becomes narrower (see inset) to compensate for the more extreme curvature near the FWHM. (b) Three different possible criteria for the resolution of two identical instrumental functions: (G) Gaussian, (L) Lorentzian, (A) Airy, (S) sinc^2, (V) Voigt.

3. Logical Dilemma

We illustrate a *reductio ad absurdum* arising from conventional definitions of spectral resolution, with the aid of a series of functions that are everywhere continuous and have the same FWHM, but have varying core to wing ratios (see Figure 1(a)). Long-slit spectrometers usually have rather Gaussian (G) profiles, whereas Fabry-Perot interferometers approach Lorentzian (L) at high finesse. The extreme wing-to-core ratio function (J) is not unlike the *spatial* response function of the *Hubble Space Telescope* prior to installation of the COSTAR optics. The Houston criterion implies that all of the profiles have equal resolving ability. In contrast, the Sparrow criterion would lead one to believe that narrow cores and large wings have better resolution capabilities (see Table 1 and Figure 1(b)). *This is clearly not physical.* A resolved core does not guarantee that two profiles are clearly separated, since the wing contribution remains unresolved. Intuitively, one way to see this is in terms of the shot noise constraint. The uncertainty in finding the line centroid depends inversely on the signal under the profile. As we increase the wing power, for a constant line flux, the signal in the core decreases dramatically.

4. Comparison of Resolution Criteria

In Table 1, we compare five different functional forms which arise in observational astronomy. Each of the functions has been expressed in terms of its FWHM δx; the Voigt function is expressed in two different ways to emphasize its Lorentzian and Gaussian behavior in the different limits. The various cases are illustrated in Figure 1(b). The figure shows that the Sparrow criterion is the least stringent, followed by the Houston criterion. The equivalent width (area) criterion is least forgiving to the Lorentzian (and therefore Airy) function. At first glance, it would appear that the Houston criterion is sufficient to resolve two lines. However, to avoid the logical dilemma described in the previous section, we should say that the lines are *properly resolved* by the area criterion.

We propose that the equivalent width (the area of the line profile divided by the peak height) is a better measure of the spectral purity of an instrumental function. This is more physical for a number of reasons: (1) the equivalent width, not the FWHM, constitutes the average width of the profile, (2) the equivalent width is more representative of the total signal under the line profile, (3) the equivalent width is a better discriminant of the wing behavior of a line profile, and (4) the equivalent width has an entirely general definition for an arbitrary positive function. For Gaussian spectrometers (e.g., long-slit devices), the correction amounts to no more than 6%. For Lorentzian spectrometers (e.g., Fabry-Perot filters), this leads to a large correction factor (50–60%) to the more standard use of the line profile FWHM.

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